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Relative Approximation

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1. INTRODUCTION

Let f be a real function, defined and $\neq 0$ on $[-1, 1]$ and let N be an integer $\geqslant 0$. Consider the problem of relative approximation of f by real polynomails $r(x)$ of degree $\leqslant N$, i.e., approximating 1 by $r(x)/f(x)$, uniformly on $[-1, 1]$. This is the same as the problem of approximating f by r in the norm

$$\sup_{-1 \leqslant x \leqslant 1} | 1/f(x)| \cdot | f(x) - r(x)| .$$

If f is continuous on $[-1, 1]$, this is just a special case of the familiar problem of uniform approximation, with a (positive, continuous) weight function, of a continuous function, by polynomials of degree $\leqslant N$, namely, the case where the weight function is the reciprocal of the approximated function.

To get away from that familiar problem we relax our assumptions. Thus, we shall assume throughout that f is defined, real, $\neq 0$, and continuous in $[-1, 1] \sim \{0\}$; k and n are given nonnegative integers, and that $x^k/f(x)$ is bounded in $[-1, 1] \sim \{0\}$.

Our aim is approximating 1, in the uniform norm, by a ratio $x^k p(x)/f(x)$, $p \in \Pi_n$, so that the measure of our approximation is

$$\sup_{x \in [-1,1] \sim \{0\}} | 1 - x^k p(x)/f(x)| , \tag{1}$$

where Π_n denotes the set of all real polynmials of degree $\leqslant n$.

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First observe that there is always a $p^* \in \Pi_n$ which minimizes (1) among all $p \in \Pi_n$. In fact, if $f(x)/x^k$ is bounded in $[-1, 1] \sim \{0\}$, then for every $p \in \Pi_n$,

$$\sup_{x\in[-1,1]\sim\{0\}} \left| 1 - \frac{x^k p(x)}{f(x)} \right| = \sup_{x\in[-1,1]\sim\{0\}} \left| \frac{x^k}{f(x)} \right| \cdot \left| \frac{f(x)}{x^k} - p(x) \right|$$

and the right-hand side has a minimum when p varies over Π_n as both the approximated function $f(x)/x^k$ and the weight $x^k/f(x)$ are bounded. If, on the other hand, $f(x)/x^k$ is unbounded, then we must have

$$\sup_{x\in[-1,1]\sim\{0\}} \left| 1 - \frac{x^k p(x)}{f(x)} \right| \geqslant 1$$

for each $p \in \Pi_n$. This in turn implies that $p(x) \equiv 0$ minimizes (1) over Π_n.

In Section 2, we investigate the questions of uniqueness and characterization of $p \in \Pi_n$ minimizing (1), while in Section 3 we study the size of the numbers in (1).

We denote by $\| \ \|$ the supremum norm over $[-1, 1] \sim \{0\}$, and set

$$\rho_n = \min_{p\in\Pi_m} \left\| 1 - \frac{x^k p(x)}{f(x)} \right\|. \tag{2}$$

2. Uniqueness and Characterization of Best Relative Approximants

We begin with a lemma showing the quantitative effect on ρ_n of a "singularity" (at 0) of $x^k/f(x)$. In what follows, set

$$\mu = \liminf_{x\to 0} \frac{x^k}{f(x)} \qquad \text{and} \qquad M = \limsup_{x\to 0} \frac{x^k}{f(x)}.$$

Both μ and M are finite due to the assumption that $x^k/f(x)$ is bounded throughout $[-1, 1] \sim \{0\}$.

Lemma 1.

I. *$\rho_n > 1$ cannot occur.*

II. *$\rho_n = 1$ if and only if $\mu \leqslant 0 \leqslant M$.*

III. *$\rho_n < 1$ if and only if $\mu > 0$ or $M < 0$.*

If $\mu > 0$ or $M < 0$ then

$$\rho_n \geqslant \left| \frac{M - \mu}{M + \mu} \right| \tag{3}$$

Proof. (I) The choice $p(x) \equiv 0$ shows that $\rho_n > 1$ is impossible. (II) Suppose that $\mu \leqslant 0 \leqslant M$. Let $p \in \Pi_n$ be such that $p(0) \geqslant 0$. Then there exists a sequence $\{x_\nu\} \subset [-1, 1] \sim \{0\}$ such that $x_\nu \to 0$ and $(x_\nu^k(x_\nu)/f(x_\nu)) \to \mu p(0)$. Thus,

$$\left\| 1 - \frac{x^k p(x)}{f(x)} \right\| \geqslant \lim_{\nu\to\infty} \left| 1 - \frac{x^k p(x_\nu)}{f(x_\nu)} \right|$$
$$= 1 - \mu p(0) \geqslant 1.$$

Similarly, if $p(0) < 0$, then using M rather than μ, we can conclude that

$$\left\| 1 - \frac{x^k p(x)}{f(x)} \right\| \geqslant 1.$$

Combining this with (I) gives the desired result. Conversely, suppose $\rho_n = 1$. By way of a contradiction, assume that $\mu > 0$. Then

$$\eta = \inf_{0<|x|\leqslant 1} \frac{x^k}{f(x)} > 0,$$

which implies

$$0 \leqslant 1 - \frac{x^k}{B \cdot f(x)} \leqslant 1 - \frac{\eta}{B} < 1,$$

where $B = \sup_{0<|x|\leqslant 1} x^k/f(x)$. Hence,

$$\rho_n \leqslant \left\| 1 - \frac{x^k}{B \cdot f(x)} \right\| < 1.$$

Similarly, the assumption $M < 0$ leads to a contradiction. Note that (III) follows from (I) and (II).

Finally, we turn to proving (3). Suppose that $M < 0$. Let $p \in \Pi_n$ satisfy $p(0) \geqslant 2/(M + \mu)$ and select a sequence $\{x_\nu\} \subset [-1, 1] \sim \{0\}$ such that $x_\nu \to 0$ and $x_\nu^k/f(x_\nu) \to M$. Then,

$$\left\| 1 - \frac{x^k p(x)}{f(x)} \right\| \geqslant \lim_{\nu\to\infty} \left| 1 - \frac{x_\nu^k p(x_\nu)}{f(x_\nu)} \right|$$
$$= | 1 - M \cdot p(0)| \geqslant 1 - M \cdot p(0)$$
$$\geqslant 1 - M \cdot \frac{2}{M + \mu} = \frac{\mu - M}{M + \mu} = \left| \frac{M - \mu}{M + \mu} \right|.$$

On the other hand, let $q \in \Pi_n$ satisfy $q(0) < 2/(M + \mu)$. Select a sequence $\{x_\nu\} \subset [-1, 1] \sim \{0\}$ such that $x_\nu \to 0$ and $x_\nu^k/f(x_\nu) \to \mu$.

Then,

$$\left\| 1 - \frac{x^k q(x)}{f(x)} \right\| \geqslant \lim_{\nu \to \infty} \left| 1 - \frac{x_\nu{}^k(x_\nu)}{f(x_\nu)} \right|$$
$$= | 1 - \mu q(0) | \geqslant \mu q(0) - 1$$
$$> \mu \cdot \frac{2}{M + \mu} - 1 = \frac{\mu - M}{M + \mu} = \left| \frac{M - \mu}{M + \mu} \right| .$$

This establishes (3). A similar argument applies if $\mu > 0$.

Next, we wish to give a sufficient condition for equality in (3). In what follows we denote, for a given $p \in \Pi_n$,

$$E(x) \equiv 1 - \frac{x^k p(x)}{f(x)} .$$

LEMMA 2. *Suppose that* $\mu > 0$ *or* $M < 0$, *and that, for a given* $p \in \Pi_n$, $\limsup_{x \to 0} E(x) = \| E \|$ *and* $\liminf_{x \to 0} E(x) = -\| E \|$. *Then p is a best relative approximant to f (i.e., p minimizes* (1) *over* Π_n) *and*

$$\rho_n = \| E \| = \left| \frac{M - \mu}{M + \mu} \right| .$$

Proof. Note that $\limsup_{x \to 0} E(x) = \max\{1 - \mu p(0), 1 - M p(0)\}$ and $\liminf_{x \to 0} E(x) = \min\{1 - \mu p(0), 1 - M p(0)\}$. Thus, always $1 - \mu p(0) = -\{1 - M p(0)\}$, implying that

$$p(0) = 2/(M + \mu) \qquad \text{and} \qquad \| E \| = |(M - \mu)/(M + \mu)|.$$

We now characterize best relative approximations via a modified alternation theorem when $\rho_n < 1$, i.e., in case (III) of Lemma 1. We say that $x_1 \in [-1, 1] \sim \{0\}$ is an extreme point of the relative approximation of f by p provided $| E(x_1) | = \| E \|$. We say that 0 is an extreme point, provided that exactly one of the equalities

$$\limsup_{x \to 0} E(x) = \| E \| , \qquad \liminf_{x \to 0} E(x) = - \| E \|$$

holds. Denote the set of these extreme points by X_p. If $\limsup_{x \to 0} E(x) = \| E \|$ and $\liminf_{x \to 0} E(x) = -\| E \|$, we shall say that 0 is a determining point. Define $\sigma(x)$ on X_p by

$$\sigma(x) = \operatorname{sgn} E(x), \qquad \text{if} \quad x \neq 0,$$
$$\sigma(0) = +1, \qquad \text{if} \quad \limsup_{x \to 0} E(x) = \| E \| ,$$
$$\sigma(0) = -1, \qquad \text{if} \quad \liminf_{x \to 0} E(x) = - \| E \| .$$

Note that if 0 is a determining point, then $0 \notin X_p$, so that $\sigma(0)$ is undefined.

THEOREM 1. *(Characterization in case $\mu M > 0$). Suppose $0 < \rho_n < 1$. Then $p \in \Pi_n$ is a best relative approximant to f if and only if either*

(a) *0 is a determining point (in which case $\| E \| = |(M - \mu)/(M + \mu)|$,*

or,

(b) *there exist $n + 2$ extreme points $-1 \leqslant x_1 < x_2 < \cdots < x_{n+2} \leqslant 1$ such that $\sigma(x_{i+1}) = -\sigma(x_i)$, $i = 1,..., n + 1$.*

Proof. Assume that (a) does not occur and that p is a best relative approximant. To show (b), suppose that $-1 \leqslant x_1 < x_2 < \cdots < x_m \leqslant 1$, with $1 \leqslant m \leqslant n + 1$, is a maximal set for which $\sigma(x_{i+1}) = -\sigma(x_i)$, if $1 \leqslant i \leqslant m - 1$ (observe that there is always at least one extreme point). Since $\rho_n > 0$, neither $E(x) \equiv \| E \|$ nor $E(x) \equiv -\| E \|$ can occur. Indeed, if the former occurred, then $\| E \| \neq 1$ (otherwise, $p(x) \equiv 0$, $\rho_n = 1$), and $x^k p(x)/((1 - \| E \|) f(x)) \equiv 1$, implying that $\rho_n = 0$. If $E(x) \equiv -\| E \|$, then $x^k p(x)/(1 + \| E \|) f(x) \equiv 1$, again implying $\rho_n = 0$. Set $t_o = -1$ and $t_m = 1$. If $m > 1$, select $\{t_i\}_{i=1}^{m-1}$, satisfying $t_o < t_1 < \cdots < t_{m-1} < t_m$, $t_i \neq 0$, $x_i < t_i < x_{i+1}$, $t_i \notin X_p$, for $i = 1,..., m - 1$, such that $\sigma(x)$ is constant on $[t_i, t_{i+1}] \cap X_p$ for $i = 0, 1,..., m - 1$. Without loss of generality, assume that $x_1 \leqslant 0$ (if not, replace $f(x)$ by $f(-x)$ and $p(x)$ by $p(-x)$). By our assumption that $\rho_n > 0$, we must have $\sigma(x_i) \neq 0$ for $i = 1,..., m$. Let us assume for convenience that $\sigma(x_1) = +1$; a similar argument will treat the case $\sigma(x_1) = -1$ but will not be given here. Define

$$p_\lambda(x) \equiv p(x) + \lambda x^k \Pi(x),$$

where $\Pi(x) = (x - t_1) \cdots (x - t_{m-1})$ if $m > 1$ and $\Pi(x) \equiv 1$ if $m = 1$, and where $\lambda \neq 0$ is a real number satisfying $\operatorname{sgn} \lambda = (-1)^{k+m-1} \operatorname{sgn} f(-1)$. We shall show that there exists such a λ for which $p_\lambda(x)$ is a better relative approximant to f than p, giving a contradiction. Consider the function

$$E_\lambda(x) = 1 - \frac{x^k p_\lambda(x)}{f(x)} = E(x) - \frac{\lambda x^k \Pi(x)}{f(x)}, \qquad x \in [-1, 1] \sim \{0\}.$$

Note that our assumption $\mu M > 0$ and continuity considerations imply that $\operatorname{sgn} y^k/f(y) = \operatorname{sgn} x^k/f(x)$ for all $x, y \in [-1, 1] \sim \{0\}$. Let s be the index for which $t_s < 0 < t_{s+1}$ and set

$$W = \{x \in [-1, t_s] \cup [t_{s+1}, 1]: | E(x)| \leqslant \| E \|/2\}.$$

Since $x^k/f(x)$ is bounded, there exists a $\delta_1 > 0$ satisfying

$$| E_\lambda(x)| = | 1 - (x^k/f(x))(p(x) + \lambda \Pi(x))| \leqslant \| E \| - \delta_1$$

for all $x \in W$, provided $|\lambda|$ is sufficiently small. Also, on each interval $[t_i, t_{i+1}]$, $i \neq s$, we may use the fact that no alternation occurs to reduce the error in the usual manner. Indeed, consider such an interval $[t_i, t_{i+1}]$, where we assume for convenience that i is even (so that $\sigma(x_{i+1}) = +1$). Thus, $E(x) > -\| E \|$ for all $x \in [t_i, t_{i+1}]$. Now, let $x \in [t_i, t_{i+1}]$ be such that $E(x) \geqslant \| E \|/2$. As observed earlier, $\operatorname{sgn}(x^k/f(x))$ is constant on $[-1, 1] \sim \{0\}$; also, $\operatorname{sgn} \Pi(x) = (-1)^{m-1}$ since i is even, so that

$$\operatorname{sgn}\left(\frac{\lambda x^k \Pi(x)}{f(x)}\right) = 1.$$

Hence,

$$E_\lambda(x) = E(x) - \frac{\lambda x^k \Pi(x)}{f(x)} < E(x).$$

Thus, by compactness, there exists $\delta_i > 0$ such that

$$-\| E \| + \delta_i \leqslant E_\lambda(x) \leqslant \| E \| - \delta_i$$

for all $x \in [t_i, t_{i+1}]$ and for all λ, with $|\lambda|$ sufficiently small. A similar argument can be given for the case when i is odd.

Finally, consider the interval $[t_s, t_{s+1}]$. Since we assume that (a) does not hold, both $\limsup_{x\to 0} E(x) = \| E \|$ and $\liminf_{x\to 0} E(x) = -\| E \|$ cannot occur simultaneously. For convenience, let us assume that $\liminf_{x\to 0} E(x) > -\| E \|$. Now if $\limsup_{x\to 0} E(x) < \| E \|$ also occurs, then for $|\lambda|$ sufficiently small,

$$-\| E \| < \limsup_{x\to 0} \left(1 - \frac{x^k p_\lambda(x)}{f(x)}\right) < \| E \|,$$

so that we can select λ as above giving a better approximation on $[t_s, t_{s+1}]$. On the other hand, suppose $\limsup_{x\to 0} E(x) = \| E \|$. In this case $\sigma(x_{s+1}) = +1$ and we may take $x_{s+1} = 0$. Also, $\lambda x^k \Pi(x)/f(x) > 0$ for $x \in (t_s, t_{s+1}) \sim 0$, as reasoned earlier. Now, since there are no negative extreme points in $[t_s, t_{s+1}]$, there exists a $\delta_4 > 0$ such that $E(x) > -\| E \| + \delta_4$ for all $x \in [t_s, t_{s+1}] \sim \{0\}$. Hence, there exists a $\delta_5 > 0$ such that $|E_\lambda(x)| = |E(x) - \lambda x^k \Pi(x)/f(x)| < \| E \| - \delta_5$ on $[t_s, t_{s+1}] \sim \{0\}$, for $|\lambda|$ sufficiently small. A similar argument can be given for the case that $\limsup_{x\to 0} E(x) < \| E \|$ and $\liminf_{x\to 0} E(x) = -\| E \|$. Collecting these results, we have that for $|\lambda|$ sufficiently small, $\| E_\lambda \| < \| E \|$, a contradiction.

Conversely, if 0 is a determining point, then by Lemma 2, p is a best relative approximant to f. Finally, assuming (a) does not hold but (b) does, we shall show that p is a best relative approximant to f. Indeed, suppose there exists a $q \in \Pi_n$ such that

$$\left\| 1 - \frac{x^k q(x)}{f(x)} \right\| \leqslant \left\| 1 - \frac{x^k p(x)}{f(x)} \right\|. \tag{4}$$

Suppose $x_i \neq 0$ is a positive extreme point; then (4) implies that

$$0 \leqslant \left(1 - \frac{x_i{}^k p(x_i)}{f(x_i)}\right) - \left(1 - \frac{x_i{}^k q(x_i)}{f(x_i)}\right) = \frac{x_i{}^k}{f(x_i)} (q(x_i) - p(x_i)).$$

Likewise, if $x_i \neq 0$ is a negative extreme point, then (4) implies that

$$0 \leqslant \left(1 - \frac{x_i{}^k q(x_i)}{f(x_i)}\right) - \left(1 - \frac{x_i{}^k p(x_i)}{f(x_i)}\right) = \frac{x_i{}^k}{f(x_i)} (p(x_i) - q(x_i)).$$

On the other hand, suppose that 0 is a positive extreme point and let $\{x_\nu\} \subset [-1, 1] \sim \{0\}$ be a sequence of points for which $x_\nu \to 0$ and $(1 - x_\nu{}^k p(x_\nu)/f(x_\nu)) \to \| E \|$. Then, since both of the sequences $\{x_\nu{}^k/f(x_\nu)\}$ and $\{1 - x_\nu{}^k q(x_\nu)/f(x_\nu)\}$ are bounded, we may extract a subsequence $\{x_\mu\}$ of $\{x_\nu\}$ for which $x_\mu{}^k/f(x_\mu) \to \beta$ and $1 - x_\mu{}^k q(x_\mu)/f(x_\mu) \to \alpha$, where $\mu \leqslant \beta \leqslant M$ and $\| E \| \geqslant \alpha$. Hence,

$$0 \leqslant \| E \| - \alpha = \lim_{\mu \to \infty} \left(\frac{x_\mu{}^k q(x_\mu)}{f(x_\mu)} - \frac{x_\mu{}^k p(x_\mu)}{f(x_\mu)}\right) = \beta(q(0) - p(0)).$$

Similarly, if 0 is a negative extreme point, we have

$$0 \leqslant \beta(p(0) - q(0)),$$

where β is defined as above. However, our assumption $\mu M > 0$, implies that $\operatorname{sgn} \beta = \operatorname{sgn}(x_i{}^k/f(x_i))$, for all $x_i \neq 0$, as reasoned earlier. From this it follows that

$$\gamma(-1)^i(p(x_i) - q(x_i)) \geqslant 0, \qquad 0, i = 1,..., n + 2,$$

where $\gamma = \pm 1$ and $\{x_i\}_{i=1}^{n+2}$ is a set of extreme points on which (b) holds. Thus, by counting multiple zeros of $p - q$ twice, we see that $p - q$ must have at least $n + 1$ zeros [2, p. 61]. Hence, $p(x) \equiv q(x)$.

THEOREM 2. *(Characterization, classification and uniqueness for general μ, M). Let $B(f)$ be the set of best relative approximants to f from Π_n.*

(I) *If $\mu \leqslant 0 \leqslant M$, then uniqueness fails.*

$$B(f) = \{p \in \Pi_n : p(x) \equiv 0, \text{ or } \operatorname{sgn} p(x) = \operatorname{sgn}(x^k/f(x)) \text{ and } |p(x)| \leqslant |2f(x)/x^k| \text{ throughout } [-1, 1] \sim \{0\}\}; \text{ and}$$
$$\rho_n = 1.$$

(II) *If $\mu > 0$ or $M < 0$, and 0 is a determining point of some best relative approximant to f, then unicity fails.*

$$B(f) = \left\{p \in \Pi_n : p(0) = \frac{2}{M + \mu} \text{ and } \frac{2\mu f(x)}{x^k(M + \mu)} \leqslant p(x) \leqslant \frac{2Mf(x)}{x^k(M + \mu)} \text{ throughout } [-1, 1] \sim \{0\}\right\}; \text{ and } \rho_n = |(M - \mu)/(M + \mu)|.$$

(III) *If $\mu > 0$ or $M < 0$ and 0 is a determining point of no best relative approximant to f, then there is a unique best relative approximant and it is characterized by* (b) *of Theorem* 1.

Proof. We omit details. In case (I), a proof that treats the subcases $\mu < 0 < M$, $\mu = 0 < M$, $\mu < 0 = M$, and $\mu = M = 0$ separately is perhaps the simplest approach. In these subcases and in case (II), the theorem follows by observing the limitations that must be imposed on p to assure $\| 1 - x^k p(x)/f(x)\| \leqslant \rho_n$. In case (III), the theorem follows from Theorem 1 part (b) where a proof of uniqueness was actually given in the last argument of the proof.

3. The Degree of Relative Approximation

In this section we consider questions concerning the degree of relative approximation. However, at the outset, let us recall that if $\mu M > 0$, then

$$\rho_n \geqslant \left| \frac{M - \mu}{M + \mu} \right|.$$

Let us assume from now on that

$$0 < A = \inf_{0<|x|\leqslant 1} \left| \frac{x^k}{f(x)} \right| \leqslant B = \sup_{0<|x|\leqslant 1} \left| \frac{x^k}{f(x)} \right| < \infty.$$

Let w be the modulus of continuity of $g(x) = f(x)/x^k$ on $0 < |x| \leqslant 1$, namely, for every $\delta \geqslant 0$, let

$$w(\delta) = \sup\{|g(x) - g(y)| : |x - y| \leqslant \delta, \quad 0 < |x| \leqslant 1, \quad 0 < |y| \leqslant 1\}.$$

Set

$$\lambda = \liminf_{x \to 0} g(x), \qquad L = \limsup_{x \to 0} g(x). \tag{5}$$

Observe that if $\mu > 0$ (as we assume henceforth),

$$B^{-1} \leqslant \lambda = M^{-1} \leqslant \mu^{-1} = L \leqslant A^{-1}.$$

Define $g(0)$ to be any number in $[\lambda, L]$. It is easy to see that now, for every $\delta \geqslant 0$,

$$w(\delta) = \sup\{| g(x) - g(y)| : | x - y | \leqslant \delta, | x | \leqslant 1, | y | \leqslant 1\}.$$

We start by mentioning the following result essentially due to Jackson, Favard, and Ahiezer–Krein (see [3, Theorem 6]).

THEOREM 3. *Let g be a real function, defined and bounded in* $[-1, 1]$, *with modulus of continuity w there. Then there exists a* $p_n \in \Pi_n$ *such that*

$$\sup_{-1 \leqslant x \leqslant 1} | g(x) - p_n(x)| \leqslant \left(1 + \frac{\pi}{4}\right) w\left(\frac{2}{n+1}\right). \tag{6}$$

Returning to our g, observe first that by (5) one can easily prove that

$$w(\delta) \geqslant L - \lambda \text{ for every } \delta > 0, \lim_{\delta \to 0^+} w(\delta) = L - \lambda. \tag{7}$$

Choose now a $p_n \in \Pi_n$, satisfying (6). If $0 < | x | \leqslant 1$, then

$$\begin{aligned} \left| \frac{f(x)}{x^k} - p_n(x) \right| &\leqslant \left(1 + \frac{\pi}{4}\right) w\left(\frac{2}{n+1}\right), \\ \left| 1 - \frac{x^k p_n(x)}{f(x)} \right| &\leqslant B\left(1 + \frac{\pi}{4}\right) w\left(\frac{2}{n+1}\right). \end{aligned} \tag{8}$$

Thus,

$$\rho_n \leqslant B\left(1 + \frac{\pi}{4}\right) w\left(\frac{2}{n+1}\right).$$

Also, by (7), $w(2/(n+1)) \geqslant L - \lambda$ and $w(2/(n+1)) \to L - \lambda$ as $n \to \infty$. Note that this is compatible with (3) since $B \geqslant M$ implies that

$$B(L - \lambda) = B \cdot \frac{M - \mu}{M \cdot \mu} \geqslant \frac{M - \mu}{\mu} > \frac{M - \mu}{M + \mu}.$$

Well-known approximating polynomials that are easy to construct are the Bernstein polynomials. Let us consider them in the present context.

To consider again our g, form the function $g(2x - 1)$, whose modulus of continuity on $[0, 1]$ is $w(2\delta)$. Let $B_n(x)$ denote the nth order Bernstein polynomial of $g(2x - 1)$. Then for an appropriate constant C (for example $C = 5/4$ (see [1, p. 20])),

$$\sup_{0 \leqslant x \leqslant 1} |g(2x - 1) - B_n(x)| \leqslant Cw\left(\frac{2}{n^{1/2}}\right).$$

If $0 < |x| \leqslant 1$, then $|g(x) - B_n((x + 1)/2)| \leqslant Cw(2/n^{1/2})$. Thus,

$$\rho_n \leqslant BCw(2/n^{1/2}).$$

Since the sequence of nth order Bernstein polynomials of a bounded function converges to it at every point of continuity, we have for every $x \in [-1, 1] \sim \{0\}$, $B_n((x + 1)/2) \to g(x)$, and so

$$x^k B_n((x + 1)/2)/f(x) \to 1.$$

Finally, observe that on closed subintervals I of $[-1, 1]$ not containing 0, and for a $p_n \in \Pi_n$

$$\max_{x \in I} \left| 1 - \frac{x^k p_n(x)}{f(x)} \right| \leqslant \left[\max_{x \in I} \left| \frac{x^k}{f(x)} \right|\right] \cdot \max_{x \in I} \left| \frac{f(x)}{x^k} - p_n(x) \right|$$

and the right-hand side can be made small to an extent depending on the smoothness of f on I, in accordance with well-known theories.

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